PHYC 511
Spring 2019

Problem Session 5 02/13/2019

(1) Problem 5.1, Jackson.

(2) Problem 5.13, Jackson.

(1) We have:

$$\vec{B} = \frac{\mu_0}{4\pi} I \oint_C d\vec{l}' \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \frac{\mu_0}{4\pi} I \oint_C d\vec{l}' \times \vec{\nabla}' \frac{1}{|\vec{x}-\vec{x}'|}$$

But, from Stoke's theorem:

$$\oint_C d\vec{l}' \cdot \vec{A} = \int_S (\vec{\nabla}' \times \vec{A}) \cdot \hat{n}' \, da'$$

Let us write $\vec{A} = \vec{V} \times \vec{e}$, where $\vec{e}$ is a constant vector. Then:

$$\oint_C d\vec{l}' \cdot (\vec{V} \times \vec{e}) = \int_S \vec{\nabla}' \times (\vec{V} \times \vec{e}) \cdot \hat{n}' \, da' = \int_S [(\vec{e} \cdot \vec{\nabla}') \vec{V} -$$

$$\vec{e}(\vec{\nabla}' \cdot \vec{V})] \cdot \hat{n}' \, da'$$

In our case, $\vec{V} = \vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$, which implies that $\vec{\nabla}' \cdot \vec{V} = 0$ for $\vec{x} \neq \vec{x}'$. We also note that:

$$(\vec{e} \cdot \vec{\nabla}') V = (\vec{e} \cdot \vec{\nabla}') \vec{\nabla}' \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = (\vec{e} \cdot \vec{\nabla}) \vec{\nabla} \left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

since $\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$

Hence:

$$\vec{e} \cdot \oint_C d\vec{l}' \times \vec{\nabla}' \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = (\vec{e} \cdot \vec{\nabla}) \int_S \left(\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) \cdot \hat{n}'\right) da'$$

Since this relation holds for any constant vector $\vec{e}$, we must have;

$$\oint_C d\vec{\ell}' \times \vec{\nabla}' \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \vec{\nabla} \int_S \left(\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\cdot \hat{n}'\right) da' = \vec{\nabla} \int \frac{-(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\cdot \hat{n}'\, da'$$

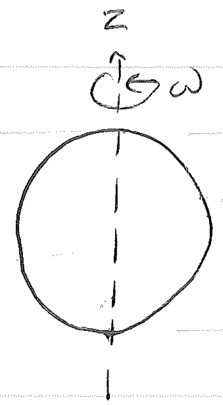
However;

$$\int_S -\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\cdot \hat{n}'\, da' = \int d\Omega' = \Omega \leftarrow$$ solid angle subtended by the loop at P

This results in:

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} I \vec{\nabla} \Omega}$$

(a) The surface current density is:



$\vec{\omega} = \omega \hat{z}$

$$\vec{K} = \sigma \vec{v} = \sigma \vec{\omega} \times \vec{x} = \sigma \omega (x\hat{y} - y\hat{x})$$

$$= \sigma \omega a \sin\theta (\cos\phi \hat{y} - \sin\phi \hat{x})$$

We have:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{x}') \, da'}{|\vec{x} - \vec{x}'|} = \frac{\mu_0 \sigma \omega a}{4\pi} \int \frac{\sin\theta' (\cos\phi' \hat{y} - \sin\phi' \hat{x})}{|\vec{x} - \vec{x}'|} a^2 \, d\Omega'$$

Both of $\sin\theta' \cos\phi'$ and $\sin\theta' \sin\phi'$ can be expressed in terms of $Y_{1,1}(\theta', \phi')$ and $Y_{1,-1}(\theta', \phi')$. Then, from the orthogonality of $Y_{\ell m}$'s, we can see that only the $\ell = 1, m = \pm 1$ terms in the multipole expansion of $\frac{1}{|\vec{x} - \vec{x}'|}$ are relevant. Thus:

$$\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a}{4\pi} \sum_{m=-1,+1} \frac{r_<}{r_>^2} \frac{4\pi}{3} \Big[Y_{1m}(\theta, \phi) \int Y_{1m}^*(\theta', \phi') \sin\theta' \cos\phi' \, d\Omega' \, \hat{y} - Y_{1m}(\theta, \phi) \int Y_{1m}^*(\theta', \phi') \sin\theta' \sin\phi' \, d\Omega' \, \hat{x} \Big]$$

Note that:

$$\sin\theta' \cos\phi' = -\frac{1}{2} \sqrt{\frac{8\pi}{3}} \left[Y_{1,1}(\theta', \phi') - Y_{1,-1}(\theta', \phi') \right], \quad \sin\theta' \sin\phi' = \frac{-1}{2i} \sqrt{\frac{8\pi}{3}}$$

$$\left[Y_{1,1}(\theta',\phi') + Y_{1,-1}^{*}(\theta',\phi')\right]$$

This leads to:

$$\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a^3}{3} \frac{r_<}{r_>^2} \left[-\sqrt{\frac{2\pi}{3}} \left(Y_{1,1}(\theta,\phi) - Y_{1,-1}(\theta,\phi) \right)\hat{y} + \frac{1}{i}\sqrt{\frac{2\pi}{3}} \left(Y_{1,1}(\theta,\phi) + Y_{1,-1}(\theta,\phi) \right)\hat{x} \right] = \frac{\mu_0 \sigma \omega a^3}{3} \frac{r_<}{r_>^2} \left[\sin\theta\cos\phi\,\hat{y} - \sin\theta\sin\phi\,\hat{x} \right]$$

$$\Rightarrow \boxed{\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a^3}{3} \frac{r_<}{r_>^2} \sin\theta\,\hat{\phi}} \qquad \left(\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right)$$

For $r<a$, we have:

$$\vec{A}(\vec{x}) = \mu_0 \sigma \omega a r \sin\theta\,\hat{\phi} \Rightarrow \boxed{\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}) = 2\mu_0 \sigma \omega a\,\hat{z}}$$

For $r>a$, we have:

$$\vec{A}(\vec{x}) = \frac{\mu_0 \sigma \omega a^4}{r^2} \sin\theta\,\hat{\phi} \Rightarrow \boxed{\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}) = \frac{2\mu_0 \omega a^4 \cos\theta}{r^3}\hat{r} + \frac{\mu_0 \sigma \omega a^4 \sin\theta}{r^3}\hat{\theta}}$$